Photon wave function and position eigenvectors

Margaret Hawton
Department of Physics, Lakehead University, Thunder Bay, ON, Canada, P7B 5E1

One and two photon wave functions are obtained by projection onto a basis of simultaneous eigenvectors of the position and number operators.

The current interest in entanglement and its application to quantum communications has rekindled the debate on the nature of the photon wave function [1, 2, 3, 4]. The photodetection amplitude has been identified with the real space photon wave function in the discussion of down conversion experiments [5], a choice that can be justified by it's relationship to photon counting which can localize the photon [6, 7]. In the standard formulation of quantum mechanics the real space wave function is the projection of the state vector onto an orthonormal basis of eigenvectors of a Hermitian position operator. However, it has been claimed since the early days of quantum mechanics that there is no position operator that defines such a basis for the photon. Here we will briefly review our recent work on the construction of a photon position operator and obtain a photon wave function by projecting onto its eigenvectors.

Attempts to arrive at a photon position operator and its associated basis of localized states go back to the early days of quantum mechanics. Pauli stated that the nonexistence of a density for the photon corresponds to the fact that the position of a photon cannot be associated with any operator in the usual sense [8]. Based on definitions of center of mass, Pryce found the **k**-space photon position operator $\widehat{\mathbf{r}}_P = i\nabla - i\mathbf{k}/2k^2 + \mathbf{k}\times\mathbf{S}/k^2$ where S_j are the 3×3 spin 1 matrices, **k** is a wave vector, and $\nabla_j = \partial/\partial k_j$ [9]. This operator does not have commuting components, and thus three spatial coordinates cannot simultaneously have a definite value. In 1949 Newton and Wigner sought *rotationally invariant* localized states and the corresponding position operators. They were successful in the case of massive particles and zero mass particles with spin 0 and 1/2, but found for photons "no localized states in the above sense exist" [10]. This result is widely quoted as a proof of the nonexistence of a photon position operator. It has been proved that there is no photon position operator with commuting components that transforms as a vector [11].

Recently we have constructed a position operator with commuting components that is not rotationally invariant [12], does not transform as a vector [13], and thus is consistent with the previous work. Description of a localized state requires a sum over all **k** and a localized photon can have definite spin in the **k**-direction, that is it can have definite helicity, but it cannot have definite spin along any fixed axis. It is the total angular momentum (AM) that has a definite value along some specified direction in space [14, 15]. The position eigenvectors are not spherically symmetric, instead they have a vortex structure as is observed for twisted light [16]. Compared to the Newton Wigner position operators for which transformation of a particle's spin and position are separable, the photon position operator must incorporate an additional unitary transfomation that reorients this vortex.

Maxwell's equations are analogous to the Dirac equation when written in terms of the Riemann-Silberstein field vector $\mathbf{F} = \mathbf{E} \pm ic\mathbf{B}$ [17, 18] where **E** and **B** are the electric and magnetic fields. This suggests that the photon is an elementary particle like any other, and that Maxwell's equations provide a first quantized description of the photon. The use of the positive frequency Riemann-Silberstein vector as a photon wave function has been thoroughly studied [19, 20]. If a field $\mathbf{\Psi}^{(1/2)}$ such as **F** that goes as $k^{1/2}$ is used as wave function, a metric factor k^{-1} is required in the scalar product. The real space squared norm then goes as $\int d^3r \int d^3r' \mathbf{\Psi}^{(1/2)*}(\mathbf{r}) \cdot \mathbf{\Psi}^{(1/2)}(\mathbf{r}') / |\mathbf{r} - \mathbf{r}'|^2$ and thus its integrand cannot be interpreted as a local number density [17]. Since the photon has no mass, it has been suggested that there is no photon number density, only energy density [20]. However, the Landau-Peierls (LP) wave function, $\mathbf{\Psi}^{(0)}$, whose absolute value squared has been interpreted as photon number density was investigated as early as 1930 [21, 22]. It has the disadvantage that its relationship to electric current density and the electromagnetic fields is nonlocal in real space [17, 23, 24]. It is possible to define a biorthonormal basis with a local scalar product that involves the eigenvectors of an operator and its adjoint [25]. This formalism has recently been applied to pseudo-Hermitian Hamiltonians that possess real spectra [26]. We will show here that such a basis provides an interesting alternative to explicit inclusion of a metric operator when working with electromagnetic fields.

In this letter one and two photon wave functions will be obtained by projection onto a basis of position operator eigenkets. Our work on the photon position operator will first be reviewed and the properties of biorthonormal bases will be outlined. The position eigenkets will be obtained in the Heisenberg picture (HP). We will then derive photon wave functions from quantum electrodynamics (QED) by projecting the state vector onto simultaneous eigenkets of the photon position operator and the number operator. Finally we will discuss the relationship of these projections to the photodetection amplitude and other real space descriptions of the photon state in the

recent literature.

We start with a discussion of the photon position operator. A **k**-space position operator with commuting components and transverse eigenvectors in the spherical polar $\widehat{\theta}$ and $\widehat{\phi}$ directions was introduced in Ref. [12]. It was generalized in Ref. [13] to allow for rotation about **k** through the Euler angle $\chi(\theta,\phi)$ to give $\widehat{\mathbf{r}}^{(\alpha,\chi)} = D\left(k^{\alpha} i\nabla k^{-\alpha}\right) D^{-1}$ where $D = \exp\left(-i\mathbf{S}\cdot\widehat{\mathbf{k}}\chi\right)\exp\left(-iS_3\phi\right)\exp\left(-iS_2\theta\right)$. The unitary transformation D rotates **k** from the z-axis to an orientation described by the angles θ and ϕ, while the transverse vectors $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ are rotated first to $\widehat{\theta}$ and $\widehat{\phi}$ and then about **k** through χ to give the unit vectors

$$\mathbf{e}^{(\chi)}_{\mathbf{k},\lambda} = \exp\left(-i\lambda\chi\right)\left(\widehat{\theta} + i\lambda\widehat{\phi}\right)/\sqrt{2}$$

with helicity $\lambda = \pm 1$. The similarity transformation k^{α} results in eigenkets proportional to k^{α} where we are interested in $\alpha = 0$ and $\pm 1/2$ as discussed above. The **k**-space position operator,

$$\widehat{\mathbf{r}}^{(\alpha,\chi)} = i\nabla - i\alpha\frac{\mathbf{k}}{k^2} + \frac{\mathbf{k}\times\mathbf{S}}{k^2} - \frac{\mathbf{k}\cdot\mathbf{S}}{k^2}\left(\widehat{\phi}\cot\theta - \nabla\chi\right), \quad (1)$$

has transverse 3-vector eigenkets satisfying

$$\widehat{\mathbf{r}}^{(\alpha,\chi)}\psi^{(\alpha)}_{\mathbf{r}_1,\lambda_1,j}(\mathbf{k}) = \mathbf{r}_1\psi^{(\alpha)}_{\mathbf{r}_1,\lambda_1,j}(\mathbf{k}) \quad (2)$$

for a photon with helictiy λ_1 at $\mathbf{r}_1$. In Eq. (2) the functional dependence distinguishes **k**-space from **r**-space, while subscripts denote eigenvalues and Cartesian components. There is a remarkable analogy between the last term in this **k**-space position operator and the **r**-space vector potential of a magnetic monpole where the Euler angle χ corresponds to a change of gauge. This was explored in Ref. [13]. It turns out that the analogy is primarily mathematical, and the last term in Eq.(1) does change the physics. The spin and orbital AM of a photon are not separable [14]. However, the z-component of the total AM operator commutes with the position operator and and this allows $\widehat{\mathbf{r}}^{(\alpha,\chi)}$, the helicity operator $\widehat{\mathbf{k}}\cdot\mathbf{S}$, $J_z = \hbar\left(-i\mathbf{p}\times\nabla + \mathbf{S}\right)$ to have simultaneous eigenvectors with eigenvalues $\mathbf{r}_1$, λ_1, and $\hbar j_z$ for integral j_z. The the first three terms of $\widehat{\mathbf{r}}^{(1/2,\chi)}$ are the Pryce position operator, $\widehat{\mathbf{r}}_P$, whose components do not commute. The last term gives the position operator commuting components, dictates that $\widehat{\mathbf{r}}^{(\alpha,\chi)}$ transform as a vector only for rotations about the z-axis as can be seen from Eq.(67) of Ref. [13], and fixes j_z for a given $\widehat{\mathbf{r}}^{(\alpha,\chi)}$. The quantum numbers $\{\mathbf{r}_1,\lambda_1\}$ index the basis states for a given j_z.

A biorthonormal basis of one photon position eigenkets will now be obtained. For $\alpha = 0$ the operator $\widehat{\mathbf{r}}^{(0,\chi)}$ is self adjoint, has real eigenvalues, and defines a single orthonormal basis as is usual in quantum mechanics. For fields, $\alpha = 1/2$ and the position operator is not self-adjoint, rather it is pseudo-Hermitian. The biorthonormal pairs, $\{\psi_n,\phi_n\}$, of eigenkets of a pseudo-Hermitian operator and its adjoint satisfy [25, 26]

$$\begin{aligned}
\widehat{O}\left|\psi_n\right\rangle = O_n\left|\psi_n\right\rangle,\ \widehat{O}^{\dagger}\left|\phi_n\right\rangle = O_n^{*}\left|\phi_n\right\rangle, \quad (3)\\
\widehat{O}^{\dagger} = \eta\widehat{O}\eta^{-1},\ \left\langle\phi_n|\psi_m\right\rangle = \delta_{n,m},\\
\sum_n\left|\psi_n\right\rangle\left\langle\phi_n\right| = \sum_n\left|\phi_n\right\rangle\left\langle\psi_n\right| = 1,
\end{aligned}$$

where η is a metric operator. If $\rho = \sqrt{\eta}$ is the positive square root of η, then $\widehat{o} = \rho\widehat{O}\rho^{-1}$ is Hermitian. To apply this formalism to the photon we take $\eta = k$ and $\alpha = -1/2$. Then $\widehat{o} = \widehat{\mathbf{r}}^{(0,\chi)}$ is Hermitian and the eigenvectors of $\widehat{O} = \widehat{\mathbf{r}}^{(-1/2,\chi)}$ and $\widehat{O}^{\dagger} = \widehat{\mathbf{r}}^{(1/2,\chi)}$ form a biorthogonal pair that go as $1/\sqrt{k}$ and $\sqrt{k}$ as required by QED for the vector potential and the electromagnetic fields respectively. Eqs. (3) then give Eq. (2) and

$$\begin{aligned}
\widehat{\mathbf{r}}^{(-1/2,\chi)\dagger} = k\widehat{\mathbf{r}}^{(-1/2,\chi)}k^{-1} = \widehat{\mathbf{r}}^{(1/2,\chi)},\\
\sum_j\left\langle\psi^{(-\alpha)}_{\mathbf{r}_2,\lambda_2,j}|\psi^{(\alpha)}_{\mathbf{r}_1,\lambda_1,j}\right\rangle = \delta^3\left(\mathbf{r}_1 - \mathbf{r}_2\right)\delta_{\lambda_1,\lambda_2},\\
\sum_{\lambda,j}\int d^3r\left|\psi^{(\alpha)}_{\mathbf{r},\lambda,j}\right\rangle\left\langle\psi^{(-\alpha)}_{\mathbf{r},\lambda,j}\right| = 1 \quad (4)
\end{aligned}$$

where δ^3 is the 3-dimensional Dirac δ-function and we can interchange α with $-\alpha$. By multiplying Eq. (2) for $\widehat{\mathbf{r}}^{(0,\chi)}$ by $\rho^{\mp 1}$ where $\rho = k^{1/2}$ and inserting $\rho^{\pm 1}\rho^{\mp 1} = 1$ between $\widehat{\mathbf{r}}$ and ψ to obtain the $\widehat{\mathbf{r}}^{(\pm 1/2,\chi)}$ eigenvector equations it can be proved that the real eigenvectors, $\mathbf{r}_1$, are preserved by the similarity transfomation to the biorthogonal basis.

The time dependence is determined by the Hamiltonian $\widehat{H} + \widehat{H}_0$ with $\widehat{H} = \sum_{\mathbf{k},\lambda}\hbar kc a^{\dagger}_{\mathbf{k},\lambda}a_{\mathbf{k},\lambda}$ where the zero point terms $\widehat{H}_0 = \sum_{\mathbf{k},\lambda}\hbar kc/2$ which are unaffected by the photon state will be omitted here. The operator $a_{\mathbf{k},\lambda}$ annihilates a photon with wave vector **k** and helicity λ. The operators and their eigenkets are time dependent in the HP [27]. Using the unitary time evolution operator $U(t) = \exp\left(-i\widehat{H}t\right)$, the position operator, given by Eq. (1) in the Schrödinger picture, becomes $\widehat{\mathbf{r}}^{(\alpha,\chi)}_{HP} = U^{\dagger}(t)\,\widehat{\mathbf{r}}^{(\alpha,\chi)}U(t)$ in the HP with eigenkets $U^{\dagger}(t)\left|\mathbf{r}_1,\lambda_1\right\rangle$ given by

$$\psi^{(\alpha)}_{\mathbf{r}_1,\lambda_1,j}(\mathbf{k},t) = k^{\alpha}e_{\mathbf{k},\lambda_1,j}\exp\left(-i\mathbf{k}\cdot\mathbf{r}_1 + ikct\right)/\sqrt{V} \quad (5)$$

in the **k**-space representation. Equivalently we can describe the 1-photon position eigenkets by defining the operators

$$\widehat{\psi}^{(\alpha)}_{\mathbf{r}_1,\lambda_1,j}(t) \equiv \sum_{\mathbf{k}} k^{\alpha}e_{\mathbf{k},\lambda_1,j}a^{\dagger}_{\mathbf{k},\lambda_1}\exp\left(-i\mathbf{k}\cdot\mathbf{r}_1 + ikct\right)/\sqrt{V} \quad (6)$$

and the kets

$$\left|\psi^{(\alpha)}_{\mathbf{r}_1,\lambda_1,j}(t)\right\rangle = \widehat{\psi}^{(\alpha)}_{\mathbf{r}_1,\lambda_1,j}(t)\left|0\right\rangle \quad (7)$$

where $\left|0\right\rangle$ is the vacuum state. The field operators are $\widehat{\mathbf{E}} = -\partial\widehat{\mathbf{A}}/\partial t$ and $\widehat{\mathbf{B}} = \nabla\times\widehat{\mathbf{A}}$ where the vector potential

operator in the Coulomb gauge can be written as $\widehat{\mathbf{A}}_{\mathbf{r}}(t) = \left[\widehat{\mathbf{A}}_{\mathbf{r}}^{(+)}(t) + \widehat{\mathbf{A}}_{\mathbf{r}}^{(-)}(t)\right]/\sqrt{2}$ with

$$\widehat{\mathbf{A}}_{\mathbf{r}}^{(+)}(t) = \mathcal{C}\sum_{\mathbf{k},\lambda} k^{-1/2}\mathbf{e}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}\exp\left(i\mathbf{k}\cdot\mathbf{r} - ikct\right)/\sqrt{V}$$

where $\widehat{\mathbf{A}}_{\mathbf{r},\lambda}^{(-)}(t) = \widehat{\mathbf{A}}_{\mathbf{r},\lambda}^{(+)\dagger}(t)$, $\mathcal{C} = \sqrt{\hbar/c\epsilon_0}$, and ϵ_0 the permittivity and c the speed of light in vacuum. The 1-photon operators given by Eq.(6) are simply related to the vector potential and electric field operators through $\widehat{\psi}_{\mathbf{r}_1,\lambda_1,j}^{(-1/2)}(\mathbf{k},t) = \widehat{A}_{\mathbf{r}_1,\lambda_1,j}^{(-)}/\mathcal{C}$ and $\widehat{\psi}_{\mathbf{r}_1,\lambda_1,j}^{(1/2)}(\mathbf{k},t) = \widehat{E}_{\mathbf{r}_1,\lambda_1,j}^{(-)}/(ic\mathcal{C})$.

A general state vector in which the number of photons and their wave vectors are uncertain can be expanded as

$$|\Psi\rangle = c_0|0\rangle + \sum_{\mathbf{k},\lambda} c_{\mathbf{k},\lambda}a_{\mathbf{k},\lambda}^{\dagger}|0\rangle + \frac{1}{2!}\sum_{\mathbf{k},\lambda;\mathbf{k}',\lambda'}\sqrt{\mathcal{N}_{\mathbf{k},\lambda;\mathbf{k}',\lambda'}}c_{\mathbf{k},\lambda;\mathbf{k}',\lambda'}a_{\mathbf{k},\lambda}^{\dagger}a_{\mathbf{k}',\lambda'}^{\dagger}|0\rangle + .. \quad (8)$$

where $c_0 = \langle 0|\Psi\rangle$, $c_{\mathbf{k},\lambda} \equiv \langle 0|a_{\mathbf{k},\lambda}|\Psi\rangle$, $c_{\mathbf{k},\lambda;\mathbf{k}',\lambda'} \equiv c_{\mathbf{k}',\lambda';\mathbf{k},\lambda} = \langle 0|a_{\mathbf{k},\lambda}a_{\mathbf{k}',\lambda'}|\Psi\rangle$, and $\mathcal{N}_{\mathbf{k},\lambda;\mathbf{k}',\lambda'} = 1 + \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}$. Division by 2! corrects for identical states obtained when the $\{\mathbf{k},\lambda\}$ subscripts are permuted while $\sqrt{\mathcal{N}}/2$ normalizes doubly occupied states. The *one photon real space wave function*, equal to the projection of this state vector onto an eigenket of $\widehat{\mathbf{r}}_{HP}^{(\alpha,\chi)}$ is

$$\Psi_j^{(\alpha)}(\mathbf{r},t) = \sum_{\lambda}\left\langle \psi_{\mathbf{r},\lambda,j}^{(\alpha)}|\Psi\right\rangle = \sum_{\mathbf{k},\lambda} c_{\mathbf{k},\lambda}\psi_{\mathbf{r},\lambda,j}^{(\alpha)*}(\mathbf{k},t) = \sum_{\mathbf{k},\lambda} c_{\mathbf{k},\lambda}e_{\mathbf{k},\lambda,j}^{*}k^{\alpha}\exp\left(i\mathbf{k}\cdot\mathbf{r} - ikct\right)/\sqrt{V} \quad (9)$$

where we have used Eqs. (7), (6) and (8). If $\alpha = 0$ this is the LP wave function, $\mathbf{\Psi}^{(0)}(\mathbf{r},t)$. The vector potential

$$\mathbf{A}^{(+)}(\mathbf{r},t) = \mathcal{C}\mathbf{\Psi}^{(-1/2)}(\mathbf{r},t)$$

determines the positive frequency fields

$$\mathbf{E}^{(+)}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{A}^{(+)}(\mathbf{r},t) = ic\mathcal{C}\mathbf{\Psi}^{(1/2)}(\mathbf{r},t),$$
$$\mathbf{B}^{(+)}(\mathbf{r},t) = \nabla\times\mathbf{A}^{(+)}(\mathbf{r},t),$$

which satisfy Maxwell's equations. The photodetection wave function is $\mathbf{E}^{(+)}(\mathbf{r},t) = \left\langle 0\left|\widehat{\mathbf{E}}^{(+)}(\mathbf{r},t)\right|\Psi\right\rangle$ [6]. The scalar product

$$\langle\Psi|\Psi\rangle = \sum_j\int d^3r\Psi_j^{(-\alpha)*}(\mathbf{r},t)\,\Psi_j^{(\alpha)}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda}|c_{\mathbf{k},\lambda}|^2 \equiv |c_1|^2$$

has a local integrand and $|c_1|^2$ is the probability for 1-photon in state $|\Psi\rangle$.

In $\mathbf{k}$-space the 1-photon LP and field wave functions identically predict probability $|c_{\mathbf{k},\lambda}|^2$ to measure momentum $\hbar\mathbf{k}$. In real space the LP wave function leads to a positive definite density $\left|\mathbf{\Psi}^{(0)}(\mathbf{r},t)\right|^2$. When using fields $\mathbf{\Psi}^{(0)}$ is replaced with the biorthonormal pair $\left\{\mathbf{\Psi}^{(1/2)},\mathbf{\Psi}^{(-1/2)}\right\}$. States with definite photon energy or angular momentum can have a definite k [14], and the relationship between their description in the LP and the biorthogonal bases is trivial, even in real space. However, for position eigenkets and real space wave functions in general these two bases are not so simply related. According to the competeness relation in Eq.(4) $\alpha = 1/2$ and $-1/2$ can be interchanged and the two options averaged to give the real density

$$n(\mathbf{r},t) = \mathrm{Re}\left\{\mathbf{\Psi}^{(1/2)*}(\mathbf{r},t)\cdot\mathbf{\Psi}^{(-1/2)}(\mathbf{r},t)\right\} = \mathrm{Re}\left\{i\epsilon_0\mathbf{E}^{(-)}(\mathbf{r},t)\cdot\mathbf{A}^{(+)}(\mathbf{r},t)/\hbar\right\}$$

which is local but *not positive definite*, and thus it is not a true probability density. This can be seen from the following example: If $|\Psi\rangle$ is a 1-photon state that includes only wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ both with helicity λ where $c_{\mathbf{k}_1,\lambda} = c_{\mathbf{k}_2,\lambda} = 1/\sqrt{2}$ then

$$n(\mathbf{r},t) = \frac{1}{2V}\left\{2 + \left(\sqrt{\frac{k_1}{k_2}} + \sqrt{\frac{k_2}{k_1}}\right)\times\cos\left[(\mathbf{k}_1 - \mathbf{k}_2)\cdot r - (k_1 - k_2)\,ct\right]\right\}.$$

The cosine term can exceed the spatially uniform time independent terms due to the $\sqrt{k}$ factors, leading to negative values. It gives zero if an integral over all space or all time is performed, explaining why the scalar product is unaffected by the similarity transformation. In an experiment that integrates over a long enough time or a large enough spatial volume, use of $\mathbf{\Psi}^{(0)}$ and the biorthonormal pair $\mathbf{\Psi}^{(\pm 1/2)}$ is equivalent. The density $i\epsilon_0\mathbf{E}^{(-)}\cdot\mathbf{A}^{(+)}/2\hbar + c.c$ has appeared before. The classical linear and angular field momenta can be written as $\sum_j\int d^3r i\epsilon_0 E_j\widehat{O}A_j/\hbar$ [14]. This form can be applied to optical beam AM calculations within the paraxial approximation [28]. The number operator $\widehat{n} = i\epsilon_0\sum_j\widehat{E}_j^{(-)}\widehat{A}_j^{(+)}/2\hbar + h.c.$ transforms as the zeroth component of a four-vector and satisfies a continuity equation [29]. The one photon density $n(\mathbf{r},t)$ equals $\langle\Psi|\widehat{n}|\Psi\rangle$ and its integral over all space is time independent consistent with the pair $\left\{\mathbf{\Psi}^{(1/2)},\mathbf{\Psi}^{(-1/2)}\right\}$ forming a basis as implied by Eqs. (4). Action, which is of this form, has appeared in calculations of laser linewidth [30].

For two photons we can project $|\Psi\rangle$ onto the 2-photon real space basis $\widehat{\psi}_{\mathbf{r},\lambda,i}^{(\alpha)}(t)\,\widehat{\psi}_{\mathbf{r}',\lambda',j}^{(\alpha)}(t')|0\rangle$. Use of Eq.(6) and $\left[a_{\mathbf{k},\lambda},a_{\mathbf{k}',\lambda'}^{\dagger}\right] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}$ then gives the correlation am-

plitude

$$\Psi_{i,j}^{(\alpha)}(\mathbf{r},\mathbf{r}',t,t') = \frac{1}{2!V}\sum_{\mathbf{k},\lambda;\mathbf{k}',\lambda'}\sqrt{\mathcal{N}_{\mathbf{k},\lambda;\mathbf{k}',\lambda'}}c_{\mathbf{k},\lambda;\mathbf{k}',\lambda'}k^{\alpha}\left(k'\right)^{\alpha} \times\Big[e_{\mathbf{k},\lambda,i}^{*}e_{\mathbf{k}',\lambda',j}^{*}e^{i\mathbf{k}\cdot\mathbf{r}-ikct}e^{i\mathbf{k}'\cdot\mathbf{r}'-ik'ct'} +e_{\mathbf{k}',\lambda',i}^{*}e_{\mathbf{k},\lambda,j}^{*}e^{i\mathbf{k}\cdot\mathbf{r}'-ikct'}e^{i\mathbf{k}'\cdot\mathbf{r}-ik'ct}\Big]. \tag{10}$$

which becomes a *two photon wave function* if we set $t' = t$. It is a *symmetric linear combination of products of one photon wave functions* in agreement with Refs. [2] and [1]. The one and two photon amplitudes given by Eqs. (9) and (10) are consistent with the use of the detection amplitude as a wave function [1, 2, 4, 5, 6, 7]. The "two-photon quantum state in coordinate space" obtained by taking the Fourier transform of the **k**-space probability amplitude in Ref. [31] is an example of recent use of the $\alpha = 0$ LP wave function. In either case, the sum over all n-photon terms for all positions and helicities provides a complete basis and thus "encodes the maximum total knowledge describing the system" as required by Schrödinger and discussed in Ref. [3]. For example, either form can be used to transform from the photon number basis to the quadrature basis if all nonzero n-photon amplitudes are known.

In summary, we have reviewed our previous work where it is demonstrated that a photon position operator does indeed exist. Because photon spin and orbital angular momentum are inseparable, its eigenvectors have a vortex structure like twisted light. We projected the QED state vectors onto simultaneous eigenvectors of this position operator and the number operator in two different ways: If all $k's$ are weighted equally the Landau-Peierls wave function is obtained. This gives a positive definite probability density, but a nonlocal relationship to fields and current sources. A biorthonormal basis consisting of eigenkets proportional to the vector potential and electric field results in a real local density, $i\epsilon_0\mathbf{E}^{(-)}\cdot\mathbf{A}^{(+)}/2\hbar + c.c$, which is not positive definite. Both of these wave functions have played a role in recent analyses of two photon entanglement [1, 2, 31]. The two photon wave function is a symmetrized product of one photon wave functions in agreement with Refs. [1] and [2]. When all photon numbers are allowed for, either basis provides a complete description of the quantum state of the electromagnetic field, equivalent to the QED state vector.

Acknowledgement: The author acknowledges the financial support of the Natural Science and Engineering Research Council of Canada.

[1] B. J. Smith and M. G. Raymer, quant-ph/0605149 (2006).
[2] G. G. Lapaire and J. E. Sipe, quant-ph/0607008 (2006).
[3] M. H. Rubin, quant-ph/0608073 (2006); T. B. Pittman, D. V.Strekalov, A. Migdall, M. H. Rubin, A. V. Sergienko, and Y. H. Shih, Phys. Rev. Lett. **77**, 1917 (1996).
[4] M. G. Raymer and B. J. Smith; A. Muthukrishnan, M. O. Scully and M. S. Zubairy, in *The Nature of Light: What is a Photon?*, edited by C. Roychoudhuri, K. Creath, A. F. Kracklauer, Proceedings of SPIE Vol. 5866 (SPIE, Bellingham, WA, 2005).
[5] S. P. Walborn, A. N. de Oliveira, R. S. Thebaldi, and C. H. Monken, Phys. Rev A **69**, 023811 (2004); Fortschr. Phys. **52** 1090 (2004).
[6] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambride Univ. Press, 1997).
[7] A. Muthukrishnan, M. O. Scully and M. S. Zubairy, Optics and Photonics News, **14**, S-18 (2003).
[8] W. Pauli, *General Principles of Quantum Mechanics* (Springer-Verlag, New York, 1980) [translation of *Prinipien der Quantentheoric,* Handbuck der Physik Vol. 24 (Springer, Berlin, 1933)].
[9] M. H. L.Pryce, Proc. Roy. Soc. London, Ser. A **195**, 62 (1948).
[10] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).
[11] T. F. Jordan, J. Math. Phys. **21**, 2028 (1980).
[12] M. Hawton, Phys. Rev. A, **59**, 954 (1999).
[13] M. Hawton and W. E. Baylis, Phys. Rev. A, **64**, 012101 (2001).
[14] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms: Introduction to Quantum Electrodynamics* (Wiley, New York, 1089), p. 50.
[15] M. Hawton and W. E. Baylis, Phys. Rev. A, **71**, 033816 (2005).
[16] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A **45**, 8185 (1992).
[17] I. Bialynicki-Birula, in *Progress in Optics XXXVI,* edited by E. Wolf (Elsevier,1996).
[18] R. H. Good, Phys. Rev. **105**, 1014 (1957).
[19] I. Bialynicki-Birula, Acta Phys. Polonica A, **86**, 97 (1994).
[20] J. E. Sipe, Phys. Rev A **52**, 1875 (1995).
[21] L. D. Landau and R. Peierls, Z. Phys. **62**, 188 (1930).
[22] A. I. Akhiezer and V. B. Beresteteskiĭ, *Quantum Electrodynamics* (Interscience Publishers, New York, 1965).
[23] E. R. Pike and S. Sarkar in *Frontiers in Quantum Optics,* edited by E.R. Pike (Adam Hilger, Briltol, 1986).
[24] R.J. Cook, Phys. Rev. A **25**. 2164 (1982).
[25] L. Fonda, G. C. Ghirardi and T. Weber, J. Math. Phys. **7**, 1643 (1966).
[26] A. Mostafazadeh, J. Math. Phys. **43**, 2814 (2002); A. Mostafazadeh and A. Batal, J. Phys. A: Math. Gen. **37**, 11645 (2004).
[27] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, 1985), p.51.
[28] S. J. van Enk and G. Nienhuis, J. Mod OPt. **41**, 963 (1994).
[29] M. Hawton and T. Melde, Phys. Rev. A **51**, 4186 (1995); M. Hawton, Phys. Rev. A, **59**, 3223 (1999).

[30] J. Arnaud, Optical and Quantum Electronics **28** , 1589 (1996).
[31] K. W. Chan, C. K. Law, and J. H. Eberly, Phys. Rev. Lett. **88**, 100402 (2002); K. W. Chan, J. P. Torres, and J. H. Eberly, quant-ph/0608163 (2006).